Finite Presentations of Hyperbolic Groups

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May 1, 2014

1 Groups into Metric Spaces

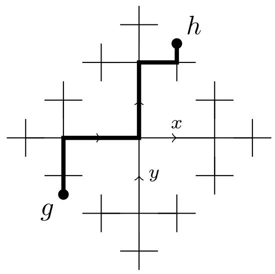
Metric spaces and the geodesics therein are absolutely foundational to geometry. The central ideas for geometric group theory are to realize algebraic objects as geometric objects, granting us the ability to pull from the rich theory of both algebra and geometry in our studies. So, we begin with the following

Definition. Let G be a group and with S the generating set. For any $g, h \in G$, define the *word metric* $d_S(g, h)$ to be the length of the shortest word representing $g^{-1}h$.

Proposition. (G, d_S) *is a metric space.*

Definition. Given a metric space (X, d) and an interval $[t_0, t_1] \subseteq \mathbb{R}$, a curve $\gamma : [t_0, t_1] \to X$ is a *geodesic* if $d(\gamma(t_0), \gamma(t_1)) = |t_0 - t_1|$ (where $|\cdot|$ is the usual absolute value on $\mathbb{R}$). We say X is a *geodesic metric space* if there exists a geodesic between every $x, y \in X$.

Although (G, d_S) is not a geodesic metric space, its Cayley graph $\Gamma_S(G)$ is, and it is the geodesics that provide the natural relationship between d_S and the Cayley graph. Let G be the vertex set on the Cayley graph. Two vertices g and h are adjacent in $\Gamma_S(G)$ precisely when $g^{-1}h \in S$ or $h^{-1}g \in S$, in which case $d_S(g, h) = 1$. Inductively, it follows that $d_S(g, h) = n$ precisely when the shortest path from g to h has length n, in which case these paths are geodesics on the Cayley graph.



The unfortunate aspect about the construction of (G, d_S) is that the properties of the metric, and thus the metric space itself, rely on the specific choice of S. While certain oddities can arise from our selection of S, their impact is somewhat limited to the finer details of the space. As such, they can often be overlooked by "viewing G from a distance."

2 Quasi-Isometries

Definition. Let (X, d) and (X', d') be metric spaces, and let $\lambda \geq 1$, $\kappa \geq 0$. A map $f : X \to X'$ is called a (λ, κ)-*quasi-isometry* if for every $x, y \in X$,

$$\frac{1}{\lambda} d(x, y) - \kappa \leq d'\left(f(x), f(y)\right) \leq \lambda d(x, y) + \kappa.$$

If there also exists some $\varepsilon > 0$ such that for every $y \in X'$, there exists a corresponding $x \in X$ for which

$$d'(y, f(x)) \leq \varepsilon$$

holds, we say that (X, d) and (X', d') are *quasi-isometric.*

Proposition. *Quasi-isometry is an equivalence relation.*

The weakening of the isometry to the quasi-isometry allows us to regard non-isometric spaces as "the same(ish)". For example, $\mathbb{Z}$ and $\mathbb{R}$ with the usual absolute value are quasi-isometric as the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1, 0)$-quasi-isometry and the map $\mathbb{R} \to \mathbb{Z}$ given by rounding to the nearest integer is a $\left(1, \frac{1}{2}\right)$-quasi-isometry. It follows then that $\mathbb{Z}^n$ and $\mathbb{R}^n$, both with the usual metrics, are quasi-isometric.

Lemma 1. *For any group G with finite generating sets S and S', (G, d_S) and $(G, d_{S'})$ are quasi-isometric.*

Proof. Let λ be the maximum length of any in $x \in S$ expressed as a word in S'. Then $\mathrm{Id} : G \to G$ is a $(\lambda, 0)$-quasi-isometry from (G, d_S) to (G, d_S) satisfying $d_{S'}(x, x) = 0$. $\square$

Because of this result, we may refer to two finitely generated groups as being quasi-isometric without ambiguity. It is natural to ask, then, which properties, if any, may be quasi-isometry invariant. The title of this paper hints at one such property.

Definition. We say that X is *δ-hyperbolic* (or just *hyperbolic*) if there exists $\delta \geq 0$ such that, for any triangle with edges that are geodesic (segments) γ_i $(i = 1, 2, 3)$ and for every $x \in \gamma_i$, then there exists $y \in \gamma_j$ $(i \neq j)$ such that $x \in B_\delta(y)$.

Lemma 2. *Let (X, d) and (X', d') be quasi-isometric geodesic metric spaces. Then (X, d) is hyperbolic if and only if (X', d') is hyperbolic.*

Before proving this, we'll need to introduce the notion of a quasi-isometric embedding of a real interval into a metric space.

Definition. Let (X, d) be a metric space. A curve $\gamma : [t_0, t_1] \to X$ is called a (λ, κ)-*quasi-geodesic* if for any subinterval $[\alpha, \beta]$ of $[t_0, t_1]$,

$$\frac{1}{\lambda}\ell\left(\gamma|_{[\alpha,\beta]}\right) - \kappa \leq d\left(\gamma(\alpha), \gamma(\beta)\right) \leq \lambda\ell\left(\gamma|_{[\alpha,\beta]}\right) + \kappa$$

where ℓ denotes the length of the curve.

Proof of Lemma 2. It suffices to show that if (X, d) is hyperbolic, then (X', d') is hyperbolic. Choose $\lambda' \geq 1$, $\kappa' \geq 0$, $\lambda = \left(\lambda'^2 + \lambda'\kappa'\right)^{-1}$, $\kappa = 2\lambda' + 3\kappa'$, and let $f : X' \to X$ be a (λ', κ')-quasi-isometry. Supposing (X, d) is hyperbolic, there exists $\delta > 0$ such that X is δ-hyperbolic. Let $x, y, z \in X'$ and consider the following geodesics (parameterized by arc length)

$$\gamma_{xy} :[0, d'(x, y)] \to X' \qquad \gamma_{xz} :[0, d'(x, z)] \to X' \qquad \gamma_{yz} :[0, d'(y, z)] \to X'.$$

The following maps are then also (λ', κ')-quasi-isometries

$$f_{xy} := f \circ \gamma_{xy} \qquad f_{xz} := f \circ \gamma_{xz} \qquad f_{yz} := f \circ \gamma_{yz}.$$

Supposing that the interval $[a, b]$ is the domain of γ_{xy}, let $D := (\mathbb{Z} \cap [a, b]) \cup \{a, b\}$ and define $\overline{\gamma}_{xy} : D \to X$ such that, for any $t_0, t_1 \in D$ with $|t_0 - t_1| \leq 1$, $\overline{\gamma}_{xy}$ maps $[n_1, n_2]$ do the geodesic segment $[f_{xy}(t_0), f_{xy}(t_1)]$.

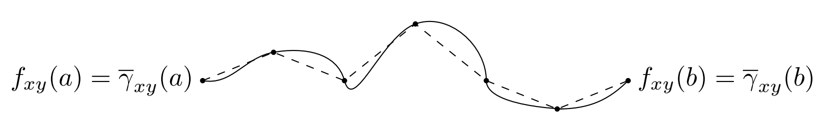


Figure 1: Quasi-geodesic segment (dashed) construction from the image of a geodesic

It is straightforward to show that $\overline{\gamma}_{xy}$ is a (λ, κ)-quasi-geodesic. From the triangle inequality of metrics, it follows that $d(f_{xy}(t), \overline{\gamma}_{xy}(t)) \leq \kappa$ for every $t \in [a, b]$.

Less obviously, it also follows (see [?, Thm. 32] for the proof) that there exists $\varepsilon > 0$ such that $\overline{\gamma}_{xy} \subseteq N_\varepsilon([f(x), f(y)])$ (the tubular ε-neighborhood), hence $[f(x), f(y)] \subseteq N_{2\varepsilon}(\overline{\gamma}_{xy})$. Therefore $[f(x), f(y)] \subseteq N_{2\varepsilon+\kappa}(f_{xy})$, so f_{xy} and $[f(x), f(y)]$ are $(2\varepsilon + \kappa)$-Hausdorff close. The analogous statements hold for both f_{xz} and f_{yz}.

Since X is hyperbolic, it follows that

$$[f(x), f(y)] \subseteq N_\delta([f(x), f(z)] \cup [f(y), f(z)]),$$

and thus $f_{xy} \subseteq N_{4\varepsilon+2\kappa+\delta}(f_{xy} \cup f_{yz})$. So, for any $t \in \gamma_{xy}$, there is some $t' \in \gamma_{xz}$ or $t' \in \gamma_{yz}$ such that $d(f(t'), f(t)) \leq 4\varepsilon+2\kappa+\delta$. As f is a (λ', κ')-quasi-isometry, it follows that $d(t, t') \leq \lambda'(4\varepsilon + 2\kappa + \delta) + \kappa'$. So, letting $\delta' = 4\varepsilon + 2\kappa + \delta$, we have that triangles in X' are δ'-thin, and therefore X' is hyperbolic. □

3 Hyperbolic Groups

We restrict our attention to groups with finite generating sets.

Definition. For a group G with generating set S, if the Cayley graph $\gamma(G, S)$ is a hyperbolic geodesic space, then G is called a *hyperbolic group.*

Hyperbolic groups do arise rather naturally. For example, every finite group is hyperbolic as the Cayley graphs are bounded. Free groups are hyperbolic as the Cayley graphs are trees, and subgroups of free groups are hyperbolic since they are also free (see [?]). More generally, finite index subgroups of hyperbolic groups are themselves hyperbolic, but as E. Rips showed in 1982, the same cannot be said about arbitrary subgroups (see [?]). The following is also a result of his (see [?, Thm. 2.3]).

Theorem. *Every hyperbolic group is finitely presented*

Proof. Let $\delta > 0$, let G be a δ-hyperbolic group, and fix some finite generating set S. Let d_S be the word metric on G. For each $k \in \mathbb{Z}^+$, define

$$\begin{aligned} B_k &:= \{g \in G : d_S(g, 1) \leq k\} \\ R_k &:= \{xyz : x, y, z \in B_k, xyz = 1 \in G\} \cup \{xx^{-1} : x \in B_k\} \\ G_k &:= \langle B_k \mid R_k \rangle. \end{aligned}$$

Since $B_1 \subseteq B_2 \subseteq \cdots$ and $R_1 \subseteq R_2 \subseteq \cdots$, we obtain the following sequence of group homomorphisms

$$G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \cdots \longrightarrow G_\infty = G$$

Our goal is to show that φ_N is an isomorphism for sufficiently large N, from which it will follow that $G = \langle B_n \mid R_n \rangle$. First, we show that each φ_k is surjective. Choose $g \in B_{k+1} \setminus B_k$, so $g = s_1 \cdots s_{k+1}$ for $s_i \in S$. Then there exist $x, y \in B_k$ (say $x = s_{k+1}^{-1}$ and $y = s_k^{-1} \cdots s_1^{-1}$) such that $xyg = 1 \in G$. Since $x, y, g \in B_{k+1}$, we have that $xyg \in R_{k+1}$.

To see injectivity, fix $N \gg 2\delta$ and suppose that $xyz \in R_{N+1}$. We aim to show that this relation can be deduced from R_N, but x, y, z need not be in B_N. To get around this, we may choose $x_1, x_2 \in B_N$ such that $x = x_1x_2$ (called a *splitting* of x) and such that $d_S(x_1, 1) > \delta$, $d_S(x_2, 1) > \delta$, and $d_S(x, 1) = d_S(x_1, 1) + d_S(x_2, 1)$ (called the *canonical splitting* of x). Then, adding the generator x and the relation $x_1x_2x^{-1}$ to the presentation for G_N, we get an equivalent presentation. From here, we now show how to deduce $xyz = 1$ from the relations in B_N.

(Case 1.) Suppose $x, y \in B_N$ and $z \in B_{N+1} \setminus B_N$. Choose a canonical splitting $z = z_1z_2$ and let P be the point corresponding to the choice of z_1 and z_2. Since the geodesic triangle Δ with vertices $1, x, xy$ is δ-thin, there exists a point Q on one of the other edges that is within δ of P - without loss of generality, suppose $Q \in [1, x]$. Then the geodesic $[P, Q]$ and the geodesic $[Q, xy]$ divide Δ into three smaller triangles, all with sides of at most length N. It follows then that the relation $xyz_1z_2 = 1$ can be deduced from three relations in R_N.

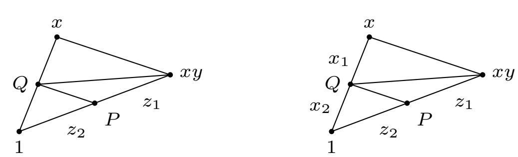


Figure 2: Partitioning Δ in Case 1 and Case 2, respectively

(Case 2.) Suppose $y, z \in B_{N+1} \setminus B_N$. By Case 1, assume all true relations of the form $abc = 1$ for $a, b \in X_N$ and $c \in X_{N+1}$. Once again, choose a canonical splitting $z = z_1z_2$ as above, and proceed similarly. Here, if Q lies on an edge of length $N+1$, then it corresponds to some splitting of either x or y - suppose $x = x_1x_2$ (which we can assume as both x_1 and x_2 have lengths at most N). Once again, we divide the triangle Δ into three smaller triangles. With this division, it's possible that one of the triangles has a single side of length $N + 1$. However, all of the other geodesic segments have length at most N, so by Case 1, we are done

Similar arguments apply to the relations of the form $xx^{-1}, x \in B_{N+1} \setminus B_N$, thus completing the proof. $\square$

References

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