Exploring Lattices in Complex Hyperbolic Space

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November 17, 2014

Motivation

In 1986, Gromov and Piatetsky-Shapiro introduced a method for constructing lattices (and in particular, nonarithmetic lattices) of all dimensions in real hyperbolic space. There is currently an open question as to whether or not an analogous technique exists for construction of lattices in complex hyperbolic space. As there are relatively few explicit computations, we know very little about what these lattices may look like.

Background

Let $\mathbb{C}^{n,1}$ denote the complex vector space $\mathbb{C}^{n+1}$ endowed with the standard Hermitian form of signature $(n, 1)$ given by

$$\langle \mathbf{w}, \mathbf{z} \rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n} - w_{n+1}\overline{z_{n+1}}.$$

We let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ be the standard projectivization map. Then *complex hyperbolic n-space* is $\mathbf{H}^n_{\mathbb{C}} := \pi(V_-)$ where $V_- = \{z \in \mathbb{C}^{n,1} : \langle z, z\rangle < 0\}$. The *boundary* of this space is $\partial\mathbf{H}^n_{\mathbb{C}} := \pi(V_0)$ where $V_0 = \{z \in \mathbb{C}^{n,1} : \langle z, z\rangle = 0\}$. As in the real hyperbolic case, we can view this space as lying in the unit n-ball in $\mathbb{C}^n$, and we call this realization the *ball model*. For $\mathbf{w}, \mathbf{z} \in V_-$, let $w = \pi(\mathbf{w})$ and $z = \pi(\mathbf{z})$. Then the metric d on $\mathbf{H}^n_{\mathbb{C}}$ is given is given by

$$\cosh^2\left(\frac{d(w, z)}{4}\right) = \frac{\langle \mathbf{w}, \mathbf{z}\rangle\langle \mathbf{z}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle\langle \mathbf{z}, \mathbf{z}\rangle}.$$

Setup

We considered the two following embeddings of $SU(1, 1, \mathbb{Z}[\omega])$ into $SU(2, 1, \mathbb{Z}[\omega])$, where $\omega = e^{2\pi i/3} = -\frac{1}{2}(1 + i\sqrt{3})$:

$$\iota_1 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}, \qquad \iota_2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

As well, for elements

$$S = \begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}, \qquad T = \begin{pmatrix} 2\omega + 2 & 2\overline{\omega} + 1 \\ 2\omega + 1 & 2\overline{\omega} + 2 \end{pmatrix},$$

and let

$$\Gamma_1 = \langle \iota_1(S), \iota_1(T)\rangle \qquad \Gamma_2 = \langle \iota_2(S), \iota_2(T)\rangle$$

Finally, let $\Gamma = \langle \Gamma_1, \Gamma_2\rangle$. It is a fact then that Γ is a *discrete subgroup* as it is a subgroup endowed with the discrete topology.

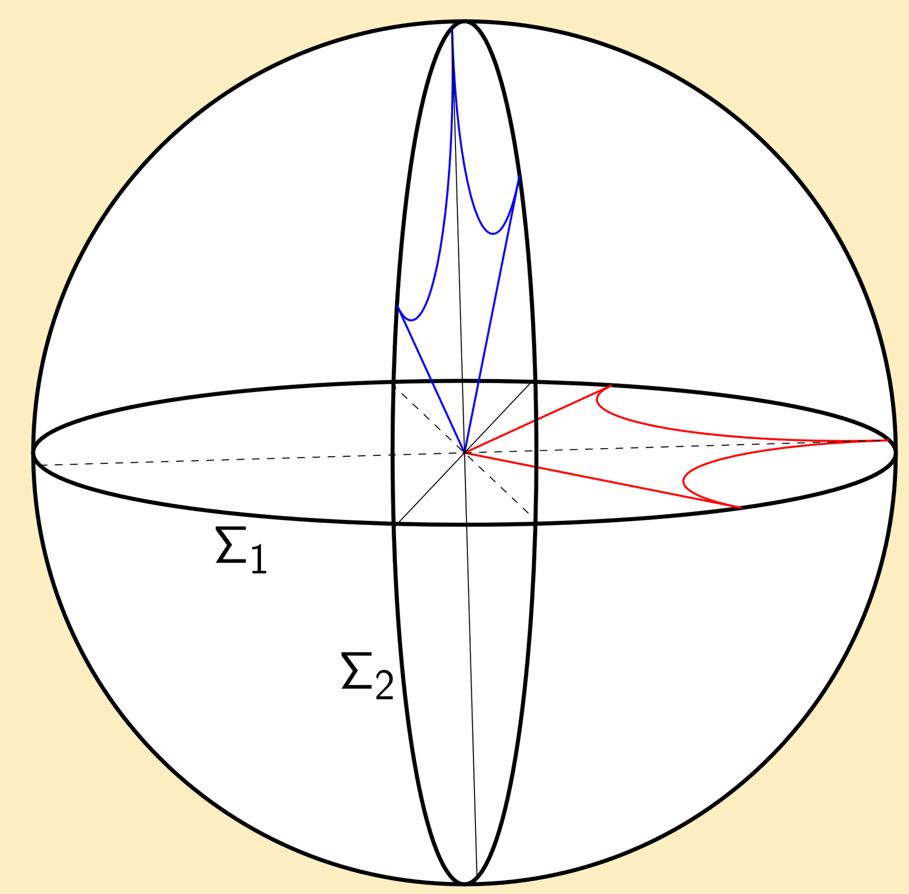
Questions

We sought to answer the following questions:

- Can we construct a fundamental domain for Γ in $\mathbf{H}^2_{\mathbb{C}}$?
- Is Γ a lattice in $SU(2, 1, \mathbb{Z}[\omega])$?
- If not, is Γ geometrically finite? In other words, does all interesting geometry occur in a compact subset?

Visualizing Complex Hyperbolic 2-Space

Since $\mathbb{C}^2$ has four real dimensions, $\mathbf{H}^2_{\mathbb{C}}$ can be viewed as sitting inside of the 4-dimensional unit ball. Here there are two copies of $\mathbf{H}^1_{\mathbb{C}}$, corresponding to each complex dimension, and these are denoted Σ_1, Σ_2.



The subgroup Γ_1 acts on Σ_1 and leaves Σ_2 invariant. Similarly, Γ_2 acts on Σ_2 and leaves Σ_1 invariant.

Fundamental Domain

Given a group G acting on a space X, a *fundamental domain* for the action is a set $D \subseteq X$ such that $X = \bigcup_{g\in G} gD$, and for distinct $g_1, g_2 \in G$, $\text{Int}(g_1D) \cap \text{Int}(g_2D) = \emptyset$. We say that G is a *lattice* if G is a discrete group and the fundamental domain has finite volume.

By explicit computation, we see that S fixes point $O = 0$, $S^{-1}T$ fixes points $P = -\overline{\omega}$, T fixes points $Q = 1$, and TS^{-1} fixes point $R = -\omega$. By identifying the interior angles and applying the Poincaré Polygon Theorem, each embedding of the quadrilateral $OPQR$ is a fundamental domain for each Γ_j, respectively.

Fundamental Domain, Cont.

The figures below demonstrate the actions of each of our generating matrices on the fundamental domain.

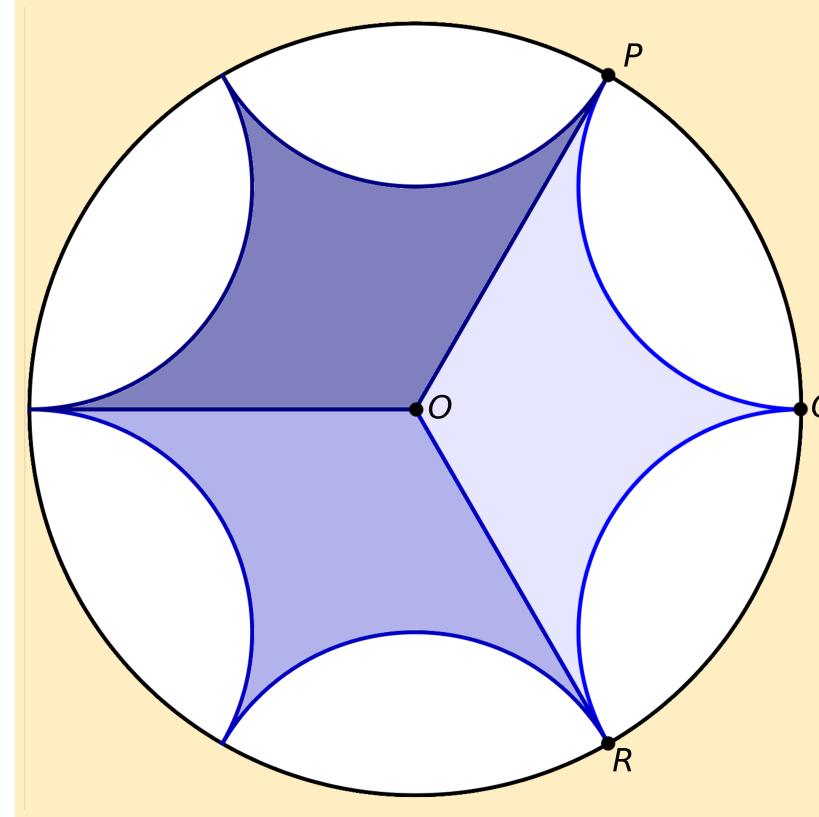


FIGURE : The action of S on the fundamental domain in $\mathbf{H}^1_{\mathbb{C}}$

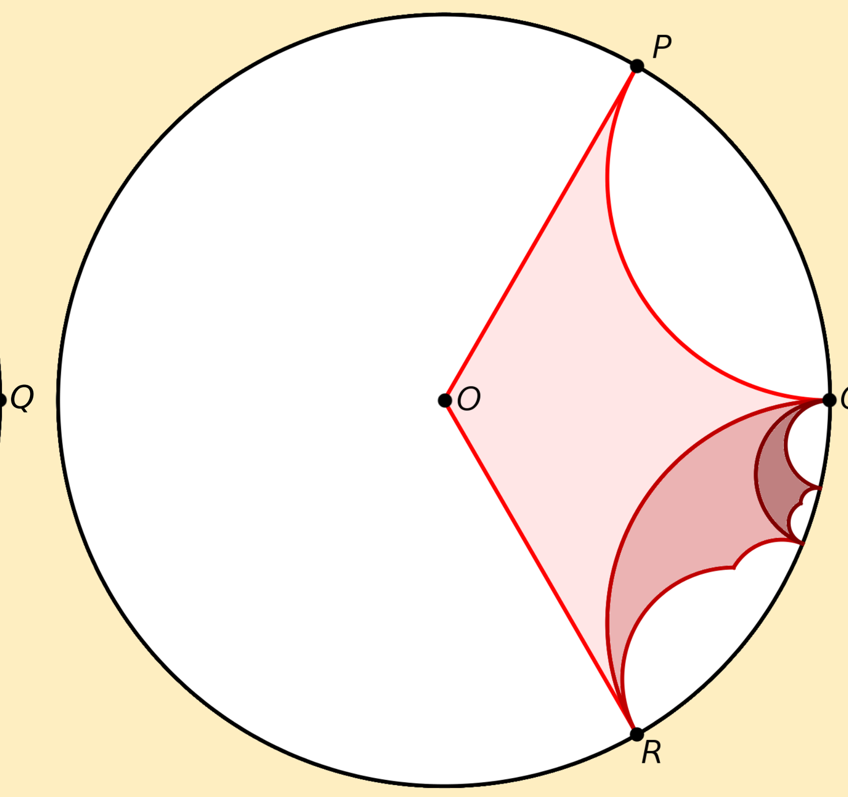


FIGURE : The action of T on the fundamental domain in $\mathbf{H}^1_{\mathbb{C}}$

Given the somewhat unpredictable nature of the action of Γ, however, it is unknown at this time whether it is feasible to compute a fundamental domain explicitly from the actions of Γ_1 and Γ_2.

What is known about the fundamental domain is that it will not have finite volume (and thus will fail to be a lattice). To see this, we let $\Pi_{\Sigma_j} : \mathbf{H}^2_{\mathbb{C}} \to \Sigma_j$ be the orthogonal projection onto Σ_j. We know that our fundamental domain must lie in $\Pi^{-1}_{\Sigma_1} \cap \Pi^{-1}_{\Sigma_2}$.

Although $\Pi^{-1}_{\Sigma_1} \cap \Pi^{-1}_{\Sigma_2}$ is 4-dimensional, but we can still attempt to visualize it and its intersection with the boundary ball $\partial\mathbf{H}^2_{\mathbb{C}}$. This intersection is nontrivial, implying that our fundamental domain has infinite volume in $\mathbf{H}^2_{\mathbb{C}}$.

References

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