A Geometric Construction of Thin Subgroups in SU(2, 1)

Joseph Wells

Arizona State University

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Review of Lattices and Thin Subgroups

For our purposes, we'll take the following as definitions.

Definition

Let $X = \mathbf{H}^n_{\mathbb{R}}$ or $\mathbf{H}^n_{\mathbb{C}}$. A discrete subgroup $\Gamma < \mathrm{Isom}^+(X)$ is a **lattice** if the X/Γ has finite volume. A subgroup $\Delta < \Gamma$ is **thin** if Δ is infinite covolume and has the same Zariski-closure as Γ.

Examples of Lattices

- $\mathbb{Z}[i]$ is a lattice in $\mathbb{C}$.
- $\mathrm{SL}(2, \mathbb{Z})$ is a lattice in $\mathrm{SL}(2, \mathbb{R})$.
- $\mathrm{SU}(2, 1; \mathcal{O}_d)$ is a lattice in $\mathrm{SU}(2, 1)$
  ($\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ where d is a positive, square-free integer). [Borel-Harish-Chandra]

Complex Hyperbolic Space

$\mathbf{H}^n_{\mathbb{C}}$ is constructed analogously to $\mathbf{H}^n_{\mathbb{R}}$.

$\mathbf{H}^n_{\mathbb{C}}$ can be viewed as the interior of the unit complex n-ball.

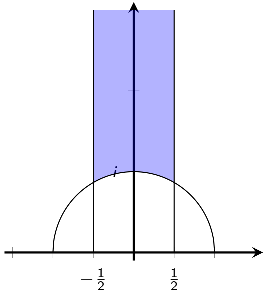
$\mathbf{H}^n_{\mathbb{C}}$ has variable negative curvature and complex structure.

$\mathbf{H}^n_{\mathbb{C}}$ has very few totally-geodesic subspaces, and none are (real) codimension-1. :-(

Initial Construction

Motivational lattice: $\mathrm{SL}(2,\mathbb{Z}) \leq \mathrm{SL}(2,\mathbb{R})$

$\mathrm{SL}(2,\mathbb{Z})$ acting on the upper half plane model of $\mathbf{H}^2_{\mathbb{R}}$



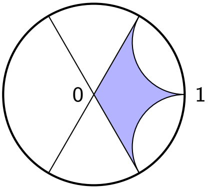
$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \langle \text{elliptic}, \text{parabolic} \rangle$$

Let ω be a third root of unity and consider the following in $\mathrm{SU}(1,1;\mathcal{O}_3)$:

$$S = \begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}, \qquad T = \begin{pmatrix} 2\omega+2 & 2\omega+1 \\ 2\overline{\omega}+1 & 2\overline{\omega}+2 \end{pmatrix}$$

$\mathrm{SU}(n,1)$ acts on $\mathbf{H}^n_{\mathbb{C}}$ by holomorphic isometries, so look at the action of $\langle S, T\rangle$ on $\mathbf{H}^1_{\mathbb{C}}$.

S is elliptic, T is parabolic. In the ball model, the fundamental domain is shown below.



$\langle S, T\rangle$ is a lattice in $\mathrm{SU}(1,1)$.

New Lattices From Old

Embed our generators S, T into $\mathrm{SU}(2,1;\mathcal{O}_3)$ so that each embedded copy of S and T fix a the subspace $\mathbf{H}^1_{\mathbb{C}}$ inside of $\mathbf{H}^2_{\mathbb{C}}$. Let Γ be the group generated by the following four matrices:

$$S_1 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 2\omega+2 & 0 & 2\omega+1 \\ 0 & 1 & 0 \\ 2\overline{\omega}+1 & 0 & 2\overline{\omega}+2 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\omega+2 & 2\omega+1 \\ 0 & 2\overline{\omega}+1 & 2\overline{\omega}+2 \end{pmatrix}.$$

Question

Is $\Gamma < \mathrm{SU}(2,1)$ a lattice?

Bisectors & Constructing Fundamental Domains

Definition

Given $z_0, z_1 \in \mathbf{H}^n_{\mathbb{C}}$, a **bisector** $B(z_0, z_1)$ is the locus of points equidistant to both z_0 and z_1.

Definition

Given $z_0 \in \mathbf{H}^n_{\mathbb{C}}$ with trivial stabilizer in Γ, **Dirichlet Domain based at** z_0 is the set

$$D_\Gamma(z_0) = \{z \in \mathbf{H}^n_{\mathbb{C}} : d(z, z_0) \leq d(\gamma z, z_0)\}.$$

Faces of $D_\Gamma(z_0)$ are contained in bisectors.

Drama

Bad news: Complex hyperbolic bisectors are terrible. Need not be totally geodesic, and any two may intersect in a disconnected set.

Constructing fundamental domains is HARD.

Hope

"Assume it is true and try to prove it. Unless it's false; then don't do that." - Julien Paupert

Question

Is Γ a thin subgroup instead?

Good news: We only care about the intersection of the fundamental group with boundary, so we switch to a view of $\mathbf{H}^2_{\mathbb{C}}$ "from infinity".

Siegel Model

$\mathbf{H}^2_{\mathbb{C}}$ analogue of the upper half space model is the Siegel model:

In $\mathbb{C} \times \mathbb{R} \times \mathbb{R}^+$, parametrize $\mathbf{H}^2_{\mathbb{C}}$ in horospherical coordinates via

$$(z, t, u) \mapsto \begin{pmatrix} -|z|^2 - u + it \\ z \\ 1 \end{pmatrix}$$

where $(z, t, 0)$ parametrizes $\partial \mathbf{H}^2_{\mathbb{C}}$, and the point at infinity is given by

$$p_\infty = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Geometry of $\partial\mathbf{H}^2_{\mathbb{C}}$

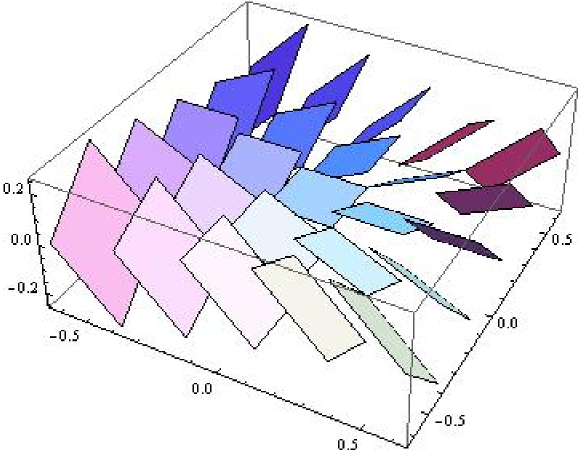


Image by Anton Lukyanenko

In this Siegel Model, the associated Hermitian form is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Apply Cayley transform to Γ generators; get conjugate subgroup $\tilde{\Gamma}$ with generators

$$\tilde{S}_1 = \begin{pmatrix} \frac{\omega}{2} + \frac{\overline{\omega}}{2} & 0 & \frac{\omega}{2} - \frac{\overline{\omega}}{2} \\ 0 & 1 & 0 \\ \frac{\omega}{2} - \frac{\overline{\omega}}{2} & 0 & \frac{\omega}{2} + \frac{\overline{\omega}}{2} \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 1 & 0 & 4\omega + 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{S}_2 = \begin{pmatrix} -\frac{1}{2}\omega & 0 & \frac{\omega}{2} + 1 \\ 0 & \omega & 0 \\ \frac{\omega}{2} + 1 & 0 & -\frac{1}{2}\omega \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} \frac{3}{2} + \overline{\omega} & \omega\sqrt{2} + \frac{1}{\sqrt{2}} & \omega + \frac{1}{2} \\ \overline{\omega}\sqrt{2} + \frac{1}{\sqrt{2}} & 2\omega + 2 & \omega\sqrt{2} + \frac{1}{\sqrt{2}} \\ \omega + \frac{1}{2} & \overline{\omega}\sqrt{2} + \frac{1}{\sqrt{2}} & \frac{3}{2} + \overline{\omega} \end{pmatrix}$$

Notably, $\tilde{T}_1$ is a vertical translation in these coordinates and fixes p_∞.

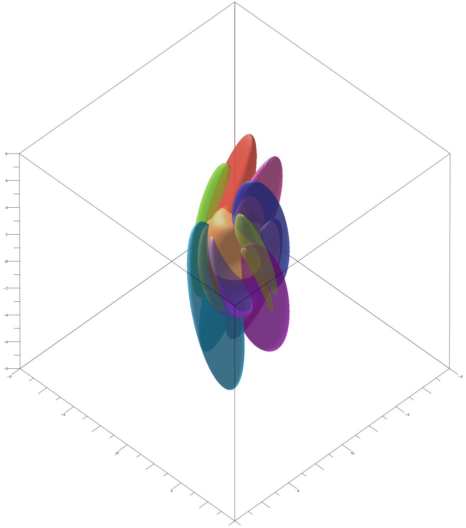
A **Ford domain** a generalization of the Dirichlet domain:

- $z_0 \in \partial \mathbf{H}^2_{\mathbb{C}}$
- Replace distance function with a Busemann function
- Bisectors called **isometric spheres**
- Ford Domain of $\tilde{\Gamma} - \mathrm{Stab}_{\tilde{\Gamma}}(z_0)$ is the common exterior of isometric spheres.

To recover a fundamental domain for Γ:

1. Find the Ford domain F for $\Gamma - \mathrm{Stab}_{\Gamma}(z_0)$.
2. Find the Dirichlet domain D for $\mathrm{Stab}_{\Gamma}(z_0)$.
3. Intersect 'em

Since $\langle \tilde{T}_1 \rangle \subseteq \mathrm{Stab}_{\tilde{\Gamma}}(p_\infty)$, we have that $\tilde{\Gamma} - \mathrm{Stab}_{\tilde{\Gamma}}(p_\infty) \subseteq \tilde{\Gamma} - \langle \tilde{T}_1 \rangle$. So, to see that Γ is thin, it suffices to see that Ford domain for $\tilde{\Gamma} - \langle \tilde{T}_1 \rangle$ has infinite volume. Experimentally, we believe this to be true.



Future Ideas/Questions

Future Ideas/Questions

a. Prove that Γ is actually a thin subgroup of the lattice $\mathrm{SU}(2,1)$.

b. Explore the combinatorics of the fundamental domain for Γ.

c. Is it true that with only slight modification, we can prove that similarly constructed finitely-generated subgroups $\mathrm{SU}(2,1;\mathcal{O}_d)$ are thin in $\mathrm{SU}(2,1)$, for all square-free positive integers d?

d. Following a recent result of Long & Reid in [LR], is there any relation between Γ and the fundamental group of a surface of genus g?

e. Since embedding a lattice subgroup Γ of $\mathrm{SU}(1,1)$ into $\mathrm{SU}(2,1)$ doesn't necessarily guarantee that we get a lattice of $\mathrm{SU}(2,1)$, what other conditions on Γ (if any) will guarantee that we *do* get a lattice in $\mathrm{SU}(2,1)$?

Thank you

Thank you.

References

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